

# Covering spheres with spheres

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## Abstract

Given a sphere of radius  $r > 1$  in an  $n$ -dimensional Euclidean space, we study the coverings of this sphere with unit spheres. Our goal is to design a covering of the lowest *covering density*, which defines the average number of unit spheres covering a point in a bigger sphere. For growing  $n$ , we obtain the covering density of  $(n \ln n)/2$ . This new upper bound is half the order  $n \ln n$  established in the classic Rogers bound.

## 1 Introduction

*Spherical coverings.* Let  $B_r^n(\mathbf{x})$  be a *ball* (solid sphere) of radius  $r$  centered at some point  $\mathbf{x} = (x_1, \dots, x_n)$  of an  $n$ -dimensional Euclidean space  $\mathbf{R}^n$ :

$$B_r^n(\mathbf{x}) \stackrel{\text{def}}{=} \left\{ \mathbf{z} \in \mathbf{R}^n \mid \sum_{i=1}^n (z_i - x_i)^2 \leq r^2 \right\}.$$

We also use a simpler notation  $B_r^n$  if a ball is centered at the origin  $\mathbf{x} = 0$ . For any subset  $A \subseteq \mathbf{R}^n$ , we say that a subset  $\text{Cov}(A, \varepsilon) \subseteq \mathbf{R}^n$  forms an  $\varepsilon$ -*covering* (an  $\varepsilon$ -net) of  $A$  if  $A$  is contained in the union of the balls of radius  $\varepsilon$  centered at points  $\mathbf{x} \in \text{Cov}(A, \varepsilon)$ . In this case, we use notation

$$\text{Cov}(A, \varepsilon) : A \subseteq \bigcup_{\mathbf{x} \in \text{Cov}(A, \varepsilon)} B_\varepsilon^n(\mathbf{x}).$$

By changing the scale in  $\mathbf{R}^n$ , we can always consider the rescaled set  $A/\varepsilon$  and the new covering  $\text{Cov}(A/\varepsilon, 1)$  with unit balls  $B_1^n(\mathbf{x})$ . Without loss of generality, below we consider these (unit) coverings. One of the classical problems is to obtain tight bounds on the covering size  $|\text{Cov}(B_r^n, 1)|$  for any ball  $B_r^n$  of radius  $r$  and dimension  $n$ .

Another related covering problem arises for a *sphere*

$$S_r^n \stackrel{\text{def}}{=} \left\{ \mathbf{z} \in \mathbf{R}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = r^2 \right\}.$$

Then a unit ball  $B_1^{n+1}(\mathbf{x})$  intersects this sphere with a spherical cap

$$C_r^n(\rho, \mathbf{y}) = S_r^n \cap B_1^{n+1}(\mathbf{x}),$$

which has some center  $\mathbf{y} \in S_r^n$ , half-chord  $\rho \leq 1$ , and the corresponding half-angle  $\alpha = \arcsin \frac{\rho}{r}$ . The biggest possible cap  $C_r^n(1, \mathbf{y})$  is obtained if the corresponding ball  $B_1^{n+1}(\mathbf{x})$  is centered at the distance

$$\|\mathbf{x}\| = \sqrt{r^2 - 1} \tag{1}$$

from the origin. To obtain a minimal covering, we shall consider the biggest caps  $C_r^n(1, \mathbf{y})$  assuming that all the centers  $\mathbf{x}$  satisfy (1).

*Covering density.* Given a set  $A$  of volume  $|A|$ , we consider any unit covering  $\text{Cov}(A, 1)$  and its minimum density

$$\delta(A) = \min_{\text{Cov}(A, 1)} \sum_{\mathbf{x} \in \text{Cov}(A, 1)} \frac{|B_1^n(\mathbf{x}) \cap A|}{|A|}.$$

Minimal coverings have been long studied for the spheres  $S_r^n$  and the balls  $B_r^n$ . In particular, the Coxeter-Few-Rogers “simplex” bound shows that for any sphere  $S_r^n$  of radius  $r \geq (1 + 1/n)^{1/2}$ ,

$$\delta(S_r^n) \geq c_1 n.$$

Here and below  $c_i$  denote some universal constants. Various upper bounds on the minimum covering density are obtained for  $B_r^n$  and  $S_r^n$  in the classic papers [3] and [4]. In particular, a sufficiently large ball  $B_r^n$  can be covered with the density

$$\delta(B_r^n) \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{5}{\ln n}\right) n \ln n. \quad (2)$$

In the recent papers [1] and [5], these estimates have been tightened for some radii  $r$  and also extended to the arbitrary radii. For a sphere  $S_r^n$  of an arbitrary radius  $r$ , the best universal estimate on the covering density known to date is obtained in [1] and [2] (see Corollary 1.2 and Remark 5.1 in [1]). This estimate gives

$$\delta(S_r^n) \leq \left(1 + \frac{2}{\ln n}\right) \left(1 + \frac{\ln \ln n}{\ln n} + \frac{\sqrt{e}}{n \ln n}\right) n \ln n. \quad (3)$$

Our main result is presented in Theorem 1, which reduces the present upper bounds about two times as  $n \rightarrow \infty$ .

**Theorem 1** *A sphere  $S_r^n$  of any radius  $r > 1$  and any dimension  $n \geq 3$  can be covered with spherical caps of half-chord 1 with density*

$$\delta(S_r^n) \leq \left(\frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{5}{\ln n}\right) n \ln n. \quad (4)$$

For  $n \rightarrow \infty$ , there exists  $o(1) \rightarrow 0$  such that

$$\delta(S_r^n) \leq \frac{1}{2} n \ln n + \left[\frac{3}{2} + o(1)\right] n \ln \ln n. \quad (5)$$

## 2 Preliminaries: embedded coverings

The approach of this section gives the estimates on  $\delta(S_r^n)$  that are similar to (3). We present this technique mainly to introduce the embedded coverings that lead to the new bound (4) in Section 3.

Consider a sphere  $S_r^n$  of some dimension  $n \geq 3$  and radius  $r > 1$ . We use notation  $C(\rho, \mathbf{y})$  for a cap  $C_r^n(\rho, \mathbf{y})$  whenever parameters  $n$  and  $r$  are fixed; we also use a shorter notation  $C(\rho)$  when a specific center  $\mathbf{y}$  is of no importance. In this case,  $\text{Cov}(\rho)$  will denote any covering of  $S_r^n$  with spherical caps  $C(\rho)$ . Let

$$\theta_\rho = \frac{|C(\rho)|}{|S_r^n|}$$

be the fraction of the surface of the sphere  $S_r^n$  covered by a cap  $C(\rho)$ . For any  $\tau < \delta \leq 1$ , we extensively employ inequality (see [1]):

$$|\theta_\tau| \geq |\theta_\delta| \left(\frac{\tau}{\delta}\right)^n. \quad (6)$$

In this section, we use parameters

$$\varepsilon = \frac{1}{n \ln n}, \quad \rho = 1 - \varepsilon. \quad (7)$$

We begin with two preliminary lemmas, which will be used to simplify our calculations.

**Lemma 2** *For any  $n \geq 4$ ,*

$$\left(1 - \frac{1}{n \ln n}\right)^{-n} < 1 + 1/\ln n + 1/\ln^2 n. \quad (8)$$

*Proof.* For  $n = 4, \dots, 8$ , the above inequality is verified numerically. For  $n \geq 9$ , we take  $z = \varepsilon + 3\varepsilon^2/2$ . Note that  $\varepsilon < 1/3$  and  $zn < 1/2$ , in which case

$$(1 - \varepsilon)^{-1} = \sum_{i=0}^{\infty} \varepsilon^i \leq 1 + z,$$

$$(1 + z)^n < \sum_{i=0}^{\infty} \frac{(zn)^i}{i!} \leq 1 + zn + 2(zn)^2/3.$$

Also,  $\ln n < 1/2$  for  $n \geq 9$ . Then  $zn \leq \ln^{-1} n + (\ln^{-2} n)/6$  and

$$(1 + z)^n < 1 + \frac{1}{\ln n} + \frac{5}{6 \ln^2 n} + \frac{1}{4 \ln^3 n},$$

which is less than the right-hand side of (8) if  $\ln n < 1/2$ . □

Let  $f_1(x)$  and  $f_2(x)$  be two positive differentiable functions. We say that  $f_1(x)$  moderates  $f_2(x)$  for  $x \geq a$  if the inequality

$$(\ln f_1(x))' \geq (\ln f_2(x))'$$

holds for all  $x \geq a$ . We will use the following simple lemma.

**Lemma 3** *Consider  $m$  functions  $f_i(x)$  such that  $f_1(x)$  moderates each function  $f_i(x)$ ,  $i \geq 2$ , for  $x \geq a$ . Then inequality*

$$f_1(x) \geq \sum_{i=2}^m f_i(x)$$

*holds for any  $x \geq a$  if it is valid for  $x = a$ .*

*Proof.* Note that  $f_i(x) = f_i(a) \exp \{s_i(x)\}$ , where

$$s_i(x) \stackrel{\text{def}}{=} \int_a^x (\ln f_i(t))' dt \leq s_1(x).$$

Therefore,

$$f_1(x) \geq \exp \{s_1(x)\} \sum_{i=2}^m f_i(a) \geq \sum_{i=2}^m f_i(a) \exp \{s_i(x)\} = \sum_{i=2}^m f_i(x),$$

which completes the proof.  $\square$

*An embedded algorithm.* To design a covering  $\text{Cov}(1)$ , we shall also use another covering

$$\text{Cov}(\varepsilon) : S_r^n \subseteq \bigcup_{\mathbf{u} \in \text{Cov}(\varepsilon)} C(\varepsilon, \mathbf{u})$$

with smaller caps  $C(\varepsilon, \mathbf{u})$ . We assume that this covering has some density  $\delta_\varepsilon$  and the corresponding size  $|\text{Cov}(\varepsilon)| = \delta_\varepsilon / \theta_\varepsilon$ . We also use parameter

$$\lambda = 1 + \frac{\ln \ln n}{\ln n} + \frac{2}{n}. \quad (9)$$

First, we randomly choose  $N$  points  $\mathbf{y} \in S_r^n$ , where

$$\frac{\lambda n \ln n}{\theta_\rho} - 1 < N \leq \frac{\lambda n \ln n}{\theta_\rho}. \quad (10)$$

Consider the set  $\{C(\rho, \mathbf{y})\}$  of  $N$  caps. Then we take all centers  $\bar{\mathbf{u}} \in \text{Cov}(\varepsilon)$  that are left uncovered by the set  $\{C(\rho, \mathbf{y})\}$  and form the extended set

$$X = \{\mathbf{y}\} \cup \{\bar{\mathbf{u}}\}.$$

By replacing the caps  $C(\rho, \mathbf{x})$ ,  $\mathbf{x} \in X$ , with the bigger caps  $C(1, \mathbf{x})$ , we obtain a covering

$$\text{Cov}(1) : S_r^n \subseteq \bigcup_{\mathbf{x} \in X} C(1, \mathbf{x}).$$

**Lemma 4** *For any  $n \geq 8$  and  $r > 1$ , a sphere  $S_r^n$  can be covered with density*

$$\delta_* \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{3}{\ln n}\right) n \ln n. \quad (11)$$

*Proof.* Consider the above covering  $X$ . Any point  $\mathbf{u}$  is covered by some cap  $C(\rho, \mathbf{y})$  with probability  $\theta_\rho$ . Therefore, we estimate the expected number  $\bar{N}$  of centers  $\bar{\mathbf{u}}$  left uncovered after  $N$  trials as follows

$$\bar{N} = (1 - \theta_\rho)^N \cdot |\text{Cov}(\varepsilon)|.$$

Here we use (6) and observe that

$$|\text{Cov}(\varepsilon)| = \delta_\varepsilon / \theta_\varepsilon \leq (n \ln n)^n \delta_\varepsilon / \theta_1.$$

To estimate  $(1 - \theta_\rho)^N$ , note that  $\theta_\rho < \theta_1 < 1/2$ . Then we use estimate (10) for  $N$  and deduce that

$$(1 - \theta_\rho)^N \leq e^{-N\theta_\rho(1+\theta_\rho/2)} \leq e^{-(\lambda n \ln n - \theta_\rho)(1+\theta_\rho/2)} \leq e^{-\lambda n \ln n},$$

$$\bar{N} \leq e^{-\lambda n \ln n} (n \ln n)^n \delta_\varepsilon / \theta_1 \leq n^{-2} \delta_\varepsilon / \theta_1.$$

According to (6) and (8),

$$\theta_1 / \theta_\rho \leq (1 - \varepsilon)^{-n} \leq 1 + 1/\ln n + 1/\ln^2 n. \quad (12)$$

Thus, there exists a random set  $X$  that gives the density

$$\delta_1 = \theta_1(N + \bar{N}) \leq (\lambda n \ln n) \theta_1 / \theta_\rho + \delta_\varepsilon / n^2 \leq \delta_* (1 - 1/n^2) + \delta_\varepsilon / n^2 \quad (13)$$

where we take

$$\delta_* = \lambda n \ln n \frac{1 + 1/\ln n + 1/\ln^2 n}{1 - 1/n^2}.$$

Now let us assume that  $\delta_1$  meets some uniform upper bound for all radii  $r$ . For example, we can use (3) or a weaker estimate  $\delta_1 \leq n^c$ , where  $c \geq 2$ . By changing the scale in  $\mathbf{R}^{n+1}$ , we can map a covering  $\text{Cov}(1)$  of any sphere  $S_r^n$  onto the covering  $\text{Cov}(\varepsilon)$  of the sphere of radius  $r\varepsilon$ . Thus,  $\delta_1$  and  $\delta_\varepsilon$  are interchangeable, and we can always use  $\delta_1$  instead of  $\delta_\varepsilon$  in the right-hand side of (13). If  $\delta_1 > \delta_*$ , this will reduce  $\delta_1$  in the left-hand side of (13). Obviously, this reduction gives the upper bound  $\delta_1 \leq \delta_*$ .

Thus, we only need to verify that  $\delta_*$  satisfies inequality (11). Here we apply Lemma 3. Then straightforward calculations show that the right-hand side of (11) exceeds  $\delta_*$  for  $n = 8$  and moderates  $\delta_*$  for  $n \geq 8$ , due to the bigger remaining term  $3/\ln n$  in (11).  $\square$

*Possible refinements.* More detailed arguments show that (11) holds for any  $n \geq 3$ , whereas for  $n \geq 7$ ,

$$\delta_* \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{2}{\ln n}\right) n \ln n.$$

To reduce this order below  $n \ln n$ , we modify our approach. Namely, the caps  $C(\rho, \mathbf{y})$  will partially cover the bigger caps  $C(\mu, \mathbf{z})$  of radius  $\mu \sim n^{-1/2}$ . However, we cannot take  $\rho = 1 - \mu$  in this design, since

$$\theta_1/\theta_{1-\mu} = \exp\{n^{1/2}\},$$

and the covering density increases exponentially, when the caps  $C(\rho, \mathbf{y})$  are expanded to  $C(1, \mathbf{y})$ . To circumvent this problem, we keep  $\rho = 1 - \varepsilon$  but use the fact that a typical cap  $C(\mu, \mathbf{z})$  is covered by *multiple caps*  $C(\rho, \mathbf{y})$ . Here we shall use only those caps  $C(\rho, \mathbf{y})$  whose centers  $\mathbf{y}$  fall within some smaller distance  $d < \rho$  to centers  $\mathbf{z}$ . Then we prove that under some restrictions, most caps  $C(\mu, \mathbf{z})$  are left with holes of radius less than  $\varepsilon$ . This approach is described in the following section.

### 3 New covering algorithm for a sphere $S_r^n$

In the sequel, we use the following parameters

$$\begin{aligned} \varepsilon &= \frac{1}{2n \ln n}, & \rho &= 1 - \varepsilon \\ \beta &= \frac{1}{2} + \frac{2 \ln \ln n}{\ln n} \\ \lambda &= \beta + \frac{5}{2 \ln n} \\ \mu &= n^{-\beta} / 2\sqrt{3} \\ d &= 1 - 2\varepsilon - \mu^2. \end{aligned} \tag{14}$$

Given any  $d \in (0, r)$ , we say that the two caps,  $C(\rho_1, \mathbf{y})$  and  $C(\rho_2, \mathbf{z})$  are *d-close* if their centers  $\mathbf{y}$  and  $\mathbf{z}$  (considered as vectors from the origin) have angle

$$\angle(\mathbf{y}, \mathbf{z}) \leq \arcsin d/r.$$

We obtain a covering  $\text{Cov}(1)$  of a sphere  $S_r^n$  with asymptotic density  $\lambda n \ln n$  as follows.

1. Let a sphere  $S_r^n$  be covered with the two different coverings  $\text{Cov}(\mu)$  and  $\text{Cov}(\varepsilon)$  :

$$\text{Cov}(\mu) : S_r^n \subseteq \bigcup_{\mathbf{z} \in \text{Cov}(\mu)} C(\mu, \mathbf{z}),$$

$$\text{Cov}(\varepsilon) : S_r^n \subseteq \bigcup_{\mathbf{u} \in \text{Cov}(\varepsilon)} C(\varepsilon, \mathbf{u}).$$

We assume that both coverings have density  $\delta_*$  of (11) or less. Then

$$|\text{Cov}(\varepsilon)| \leq \frac{\delta_*}{\theta_\varepsilon}, \quad |\text{Cov}(\mu)| \leq \frac{\delta_*}{\theta_\mu}.$$

2. Randomly choose  $N$  points  $\mathbf{y} \in S_r^n$ , where

$$\frac{\lambda n \ln n}{\theta_d} - 1 < N \leq \frac{\lambda n \ln n}{\theta_d}. \quad (15)$$

Consider  $N$  spherical caps  $C(\rho, \mathbf{y})$ .

3. Let  $\bar{\mathbf{u}} \in \text{Cov}(\varepsilon)$  be any center left uncovered by the bigger caps  $C(\rho, \mathbf{y})$ , and let  $C(\mu, \bar{\mathbf{z}})$  be any cap of the covering  $\text{Cov}(\mu)$  that contains at least one such center  $\bar{\mathbf{u}}$ . We consider the entire set  $\{\bar{\mathbf{z}}\}$  of such centers and form the joint set

$$X = \{\mathbf{y}\} \cup \{\bar{\mathbf{z}}\}.$$

Note that the centers  $\mathbf{x} \in X$  form a unit covering

$$\text{Cov}(1) : S_r^n \subseteq \bigcup_{\mathbf{x} \in X} C(1, \mathbf{x}).$$

Our goal now is to prove bound (4). To simplify our calculations, we wish to eliminate the case of small  $n$ . Here we first observe (by numerical comparison) that bound (4) or even its refined version (31) exceed bound (11) for  $n \leq 100$ . Thus, Theorem 1 holds in this case, and we can only consider dimensions  $n \geq 100$  in our proof. In addition, this latter restriction also allows us to refine the residual terms in (4). In the end of the proof, we will also address the asymptotic case  $n \rightarrow \infty$ , which is much more straightforward. The proof is based on three lemmas.

We first estimate the fraction of a cap  $C(\mu, Z)$  left uncovered by a single  $d$ -close cap  $C(\rho, Y)$ .

**Lemma 5** *Consider two  $d$ -close caps  $C(\mu, Z)$  and  $C(\rho, Y)$  in a sphere  $S_r^n$ , so that*

$$\sin \angle(Y, Z) \leq d/r, \quad d = 1 - 2\varepsilon - \mu^2. \quad (16)$$

*Then the bigger cap  $C(\rho, Y)$  covers the smaller cap  $C(\mu, Z)$ , with an exception of its fraction*

$$\frac{|C(\mu, Z) \setminus C(\rho, Y)|}{|C(\mu, Z)|} \leq \omega,$$

where

$$\omega = \frac{1}{4 \ln n} \left(1 - \frac{3}{n} \ln^2 n\right)^{\frac{n-1}{2}} \leq \frac{1}{4 \ln n} \exp\left(-\frac{3}{2} \ln^2 n\right) \quad (17)$$

*Proof.* In Fig. 1, we represent the two caps  $C(\mu, Z)$  and  $C(\rho, Y)$ . The caps have bases  $PQRS$  and  $PMRT$ , which form the balls  $B_\mu^n(A)$  and  $B_\rho^n(B)$ . The bigger cap  $C(\rho, Y)$  covers the base  $B_\mu^n(A)$  of the smaller cap, with the exception of the segment  $PQRN$ . Note that the boundary of the base  $PQRS$  is the sphere  $S_\mu^{n-1}(A)$ . In turn, the boundary of the uncovered segment  $PQRN$  forms a cap  $PQR$  on this sphere, with center  $Q$  and half-angle  $\alpha = \angle PAQ$ . We first estimate  $\alpha$ .

Let  $d(H, G)$  denote the distance between any two points  $H$  and  $G$ . Also, let  $\sigma(H)$  be the distance from a point  $H$  to the line  $OBY$  that connects the origin  $O$  of the sphere  $S_r^n$  to the center  $B$  of the bigger base  $B_\rho^n(B)$  and then to the center  $Y$  of the cap  $C(\rho, Y)$ . Then according to (16),  $\sigma(Z) \leq d$ , and thus

$$\sigma(A) \leq d.$$

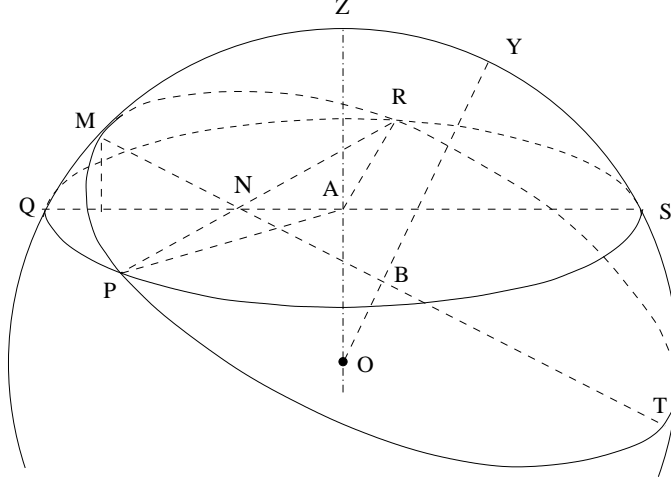


Figure 1: Two intersecting caps  $C(\mu, Z)$  and  $C(\rho, Y)$  with bases  $PQRS$  and  $PMRT$ .

Next, consider the base  $PNR$  of the uncovered cap  $PQR$ , which forms a ball in  $\mathbf{R}^{n-1}$  with center  $N$  and radius  $NP$ . Then both lines  $AN$  and  $BN$  are orthogonal to this base. Also,  $d(B, P) = \rho$ , and  $d(N, P) \leq d(A, P) = \mu$ . Thus,

$$d(B, N) = \sqrt{d^2(B, P) - d^2(N, P)} \geq \sqrt{\rho^2 - \mu^2} \geq \rho - \mu^2. \quad (18)$$

The latter inequality follows from the trivial inequality  $2\rho - 1 \geq \mu^2$ . Finally, note that by definition of the center  $B$  of the base  $PMRT$ , the lines  $BN$  and  $OBY$  are orthogonal. Then  $\sigma(N) = d(B, N)$  and

$$d(A, N) \geq \sigma(N) - \sigma(A) \geq \rho - \mu^2 - d \geq \varepsilon. \quad (19)$$

Now we consider the right triangle  $ANP$  and deduce that

$$\cos \alpha = d(A, N)/d(A, P) \geq \varepsilon/\mu = \sqrt{\frac{3}{n}} \ln n. \quad (20)$$

(Numerical calculation shows that  $\varepsilon > \mu$  for  $n \leq 41$ . This gives the trivial case, where  $\alpha = 0$  and the base  $PQRS$  is covered entirely.) Now we can calculate the fraction  $\theta$  of the boundary  $PQRS$  contained in the cap  $PQR$ . According to [1], this fraction is defined by its angle  $\alpha$  as

$$\theta < \{2\pi(n-1)\}^{-1/2} \frac{\sin^{n-1} \alpha}{\cos \alpha} < \frac{\sin^{n-1} \alpha}{4 \ln n}.$$

In the last inequality, we used the fact that  $6\pi(1 - 1/n) > 16$  for  $n \geq 7$ . Given (20), we deduce that

$$\theta \leq \frac{1}{4 \ln n} \left(1 - \frac{3}{n} \ln^2 n\right)^{\frac{n-1}{2}} \leq \frac{1}{4 \ln n} \exp \left\{ \left(-\frac{3}{2} \ln^2 n - \frac{9}{4n} \ln^4 n\right) \left(1 - \frac{1}{n}\right) \right\} \leq \frac{1}{4 \ln n} \exp \left(-\frac{3}{2} \ln^2 n\right).$$

Finally, consider any other cap  $C(\mu', Z)$  with the same center  $Z$  and a smaller half-chord  $\mu' < \mu$ . Similarly, we can define its base with some center  $A'$  on the line  $AZ$ , and its boundary  $S_{\mu'}^{n-1}(A')$ . Then we estimate the fraction  $\theta'$  of this boundary left uncovered by the bigger cap  $C(\rho, Y)$ . To obtain the upper bound on  $\theta'$ , we only need to replace  $\mu$  with  $\mu'$  in (20). This gives the angle  $\alpha' \leq \arccos(\varepsilon/\mu')$ , where again  $\alpha' = 0$  if  $\varepsilon \geq \mu'$ . Thus, we obtain a smaller uncovered fraction

$\theta' \leq \theta$  for any other boundary layer of the cap  $C(\mu, Z)$ . Then the uncovered fraction  $\omega$  of the entire cap  $C(\mu, Z)$  is also bounded by the uncovered fraction  $\theta$  of its base.  $\square$

*Remark.* The above proof is fully based on our choice of  $d$  in (16). In fact, only half the cap  $C(\mu, Z)$  is left uncovered if a cap  $C(\rho, Y)$  is  $(d + \varepsilon)$ -close. If this distance between the caps is further increased to  $\rho$ , then  $C(\rho, Y)$  covers a vanishing fraction of  $C(\mu, Z)$  as  $n \rightarrow \infty$ .

Our next goal is to prove that a typical center  $\mathbf{z}$  has sufficiently many  $d$ -close caps  $C(\rho, \mathbf{y})$  after  $N$  trials. Given any  $\mathbf{z}$ , a randomly chosen center  $\mathbf{y}$  is  $d$ -close to  $\mathbf{z}$  with the probability  $\theta_d$ . Then for any  $\mathbf{z}$ , the expected number of  $d$ -close caps  $C(\rho, \mathbf{y})$  is defined by (15):

$$\theta_d N = \lambda n \ln n - \nu, \quad \nu \in [0, \theta_d]. \quad (21)$$

We say that a center  $\mathbf{z}$  is bad and denote it  $\mathbf{z}'$  if it has only  $s$  or fewer  $d$ -close caps  $C(\rho, \mathbf{y})$ , where we take

$$s = \lfloor n/q \rfloor, \quad q = 3 \ln \ln n. \quad (22)$$

Otherwise, we say that  $\mathbf{z}$  is good and denote it  $\mathbf{z}''$ . We now estimate the expected number  $N'$  of bad centers  $\mathbf{z}'$  left after  $N$  trials.

**Lemma 6** *For  $n \geq 100$ , the expected number  $N'$  of bad centers  $\mathbf{z}'$  is*

$$N' < 2^{-n/4} N. \quad (23)$$

*Proof.* Given any center  $\mathbf{z}$ , the probability that  $\mathbf{z}$  has  $s$  or fewer  $d$ -close caps is

$$P = \sum_{i=0}^s \binom{N}{i} \theta_d^i (1 - \theta_d)^{N-i}$$

Note that for any  $i \leq s$ ,

$$(1 - \theta_d)^{N-i} \leq \exp \left\{ -(\theta_d + \theta_d^2/2)(N - i) \right\}.$$

For any  $\theta_d \leq 1/2$ , we then observe that

$$\begin{aligned} (\theta_d + \theta_d^2/2)(N - i) &= N\theta_d \left(1 + \frac{\theta_d}{2}\right) - i\theta_d \left(1 + \frac{\theta_d}{2}\right) \\ &\geq (\lambda n \ln n - \theta_d) + \frac{\theta_d}{2} [\lambda n \ln n - \theta_d - s(2 + \theta_d)] \geq \lambda n \ln n. \end{aligned}$$

To obtain the last inequality, we use (22), which yields the inequalities  $s(2 + \theta_d) \leq \frac{5}{6} \frac{n}{\ln \ln n}$  and

$$\lambda n \ln n - \theta_d - s(2 + \theta_d) \geq 2.$$

Then

$$P \leq e^{-\lambda n \ln n} \sum_{i=0}^s \frac{(\theta_d N)^i}{i!} \leq e^{-\lambda n \ln n} \sum_{i=0}^s \frac{(\lambda n \ln n)^i}{i!} \quad (24)$$

Note that consecutive summands in (24) differ at least  $(\lambda n \ln n)/s \geq \lambda q \ln n$  times. Therefore

$$\begin{aligned} \sum_{i=0}^s \frac{(\lambda n \ln n)^i}{i!} &\leq \frac{(\lambda n \ln n)^s}{s!} \sum_{i=0}^{\infty} (\lambda q \ln n)^{-i} \\ &\leq 2 \frac{(\lambda n \ln n)^s}{s!} \leq \frac{(\lambda n \ln n)^s}{(s/e)^s} \leq (e \lambda q \ln n)^{n/q}. \end{aligned}$$



Here we first estimated the sum of the geometric series as  $c_n < c_{100} < 2$ . Then we used the Stirling formula in the form  $s! > (2\pi s)^{1/2}(s/e)^s$  and removed the vanishing term  $2(2\pi s)^{-1/2}$ . Finally, the last inequality follows from the fact that its left-hand side is an increasing function of  $s$  for any  $s < \lambda n \ln n$ . Summarizing, these substitutions give

$$P \leq \exp \{nh_n - \lambda n \ln n\},$$

where

$$h_n = \frac{\ln(e\lambda q \ln n)}{q} = \frac{1}{3} + \frac{\ln(e\lambda q)}{q}.$$

Next, compare  $\delta_*$  in (11) with  $\lambda$  in (14). It is easy to verify that for any  $n$ ,

$$\delta_* \leq 2(\lambda n \ln n - 1/2) \leq 2\theta_d N.$$

Now we estimate the size of  $\text{Cov}(\mu)$  using (6) and (14) as follows

$$|\text{Cov}(\mu)| \leq \delta_*/\theta_\mu \leq 2N\theta_d/\theta_\mu \leq 2N\mu^n \leq 2N \exp \{n[\beta \ln n + \ln(12)/2]\}. \quad (25)$$

Thus, the expected number of bad caps is

$$\begin{aligned} N' &\leq |\text{Cov}(\mu)| P \leq 2N \exp \{n[h_n - (\lambda - \beta) \ln n + \ln(12)/2]\} \\ &\leq 2N \exp \{n[h_n - (5 - \ln 12)/2]\}. \end{aligned} \quad (26)$$

Now we see that the quantity  $\Psi_n$  in the brackets of (26) consists of the declining positive function  $h_n$  and the negative constant. Thus,  $\Psi_n$  is a declining function of  $n$ . Direct calculation shows that  $\Psi_{100} < -0.257$ . Therefore estimate (23) holds.  $\square$

We now consider the good centers  $\mathbf{z}''$ , and prove that  $N$  random caps  $C(\rho, \mathbf{y})$  typically leave only few  $\varepsilon$ -holes in all good caps  $C(\mu, \mathbf{z}'')$ . More precisely, let  $N''$  be the expected number of centers  $\bar{\mathbf{u}}'' \in \text{Cov}(\varepsilon)$  left uncovered in all the caps  $C(\mu, \mathbf{z}'')$ .

**Lemma 7** *For any  $n \geq 100$ , the number of uncovered centers  $\bar{\mathbf{u}}'' \in \text{Cov}(\varepsilon)$  has expectation*

$$N'' < 2^{-n/2} N.$$

*Proof.* We first estimate the total number  $|\text{Cov}(\varepsilon)|$  of centers  $\mathbf{u}$ . Similarly to (25),

$$|\text{Cov}(\varepsilon)| \leq \delta_*/\theta_\varepsilon \leq 2N\theta_d/\theta_\varepsilon \leq 2N(2n \ln n)^n.$$

Any cap  $C(\mu, \mathbf{z}'')$  is covered at least  $s+1$  times. According to Lemma 5, each covering leaves uncovered at most a fraction  $\omega$  of its surface. Therefore any point of the cap  $C(\mu, \mathbf{z}'')$  is left uncovered with probability  $\omega^{s+1}$  or less. Note that  $\omega^{s+1} < \omega^{n/q}$ , where  $q = 3 \ln \ln n$ . Then we use the upper bound (17) for  $\omega$  and deduce that

$$N'' \leq |\text{Cov}(\varepsilon)| \cdot \omega^{n/q} \leq 2N \exp \{n\Phi_n\}, \quad (27)$$

where

$$\Phi_n = \ln n + \ln \ln n - \frac{\ln 4}{3 \ln \ln n} - \frac{\ln^2 n}{2 \ln \ln n} + \ln 2 - 1/3. \quad (28)$$

Direct calculation shows that  $\Phi_{100} < -0.71$ . Note also that the first three (growing) terms in  $\Phi_n$  are moderated by the term  $(\ln^2 n)/(2 \ln \ln n)$ . Thus,  $\Phi_n < -0.71$  for all  $n \geq 100$ , and the lemma is proved.  $\square$

*Proof of Theorem 1.* Let  $\bar{\mathbf{z}}$  be any center, whose cap  $C(\mu, \bar{\mathbf{z}})$  contains at least one uncovered center  $\bar{\mathbf{u}} \in \text{Cov}(\varepsilon)$  and let  $\bar{\mathbf{z}}''$  be any such center among good centers  $\mathbf{z}''$ . Then

$$\{\bar{\mathbf{z}}\} \subseteq \{\mathbf{z}'\} \cup \{\bar{\mathbf{z}}''\}.$$

Obviously,  $|\{\bar{\mathbf{z}}''\}| \leq |\{\bar{\mathbf{u}}''\}|$ . Then, according to Lemmas 6 and 7, the set  $\{\bar{\mathbf{z}}\}$  has expected size

$$\bar{N} \leq N' + N'' < 2^{1-n/4} N.$$

(Equivalently, we can directly consider the set  $\{\mathbf{z}'\} \cup \{\bar{\mathbf{u}}''\}$ ). Thus, there exist  $N$  randomly chosen centers  $\mathbf{y}$  that leave at most  $2^{1-n/4} N$  centers  $\bar{\mathbf{z}}$ . The extended set of centers

$$X = \{\mathbf{y}\} \cup \{\bar{\mathbf{z}}\},$$

forms a 1-covering of  $S_r^n$  with caps  $\{C(1, \mathbf{x}), \mathbf{x} \in X\}$ . This covering has density

$$\delta \leq \theta_1 N (1 + 2^{1-n/4}) \leq \lambda n \ln n \left(1 + 2^{1-n/4}\right) \theta_1 / \theta_d.$$

According to inequality (8),

$$\theta_1 / \theta_d \leq \left(1 - \frac{1}{n \ln n} - n^{-2\beta}\right)^{-n} < 1 + \frac{1}{\ln n} + \frac{1}{\ln^2 n}. \quad (29)$$

Finally, we take  $\lambda$  of (8) and combine the last two inequalities as follows

$$\begin{aligned} \frac{\delta}{n \ln n} &\leq \left(\frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{5}{2 \ln n}\right) \left(1 + \frac{1}{\ln n} + \frac{1}{\ln^2 n}\right) (1 + 2^{1-n/4}) \\ &< \frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{5}{\ln n}. \end{aligned}$$

Here we again used Lemma 3. Namely, we numerically verify that the last expression exceeds the previous one for  $n = 100$  and moderates it for larger  $n$ , due to its bigger remaining term  $5/\ln n$ . This completes the non-asymptotic case  $n \geq 3$ .

To complete the proof of Theorem 1, we now present similar estimates for  $n \rightarrow \infty$ . We take any constant  $b > 3/2$  and redefine the parameters in (14) and (22) as follows:

$$\begin{aligned} \beta &= \frac{1}{2} + b \frac{\ln \ln n}{\ln n} \\ \lambda &= \beta + \frac{3}{4 \ln n} \\ \mu &= \frac{1}{2\sqrt{n} \ln^b n} \\ q &= \ln^2 \ln n. \end{aligned}$$

First, bounds (20) and (17) can be replaced with

$$\begin{aligned} \cos \alpha &\geq \varepsilon / \mu \geq \frac{\ln^{b-1} n}{\sqrt{n}} \\ \omega &\leq \left(1 - \frac{1}{n} \ln^{2b-2} n\right)^{\frac{n-1}{2}} \leq \exp\left(-\frac{1}{2} \ln^{2b-2} n\right). \end{aligned} \quad (30)$$

Second, bound (26) can be rewritten as

$$N' \leq 2N \exp \left\{ n \left( \frac{\ln(e\lambda q \ln n)}{q} + \ln 2 - \frac{3}{4} \right) \right\}.$$

Note that the first term  $\ln(e\lambda q \ln n)/q$  vanishes for  $n \rightarrow \infty$ , and  $N'$  declines exponentially in  $n$ . Thus, Lemma 6 holds. Finally, bound (27) is replaced with

$$N'' \leq 2N \exp \left\{ n \left[ \ln(2n \ln n) - \frac{\ln^{2b-2} n}{2 \ln^2 \ln n} \right] \right\}$$

which declines (faster than exponent in  $n$ ) for any given  $b > 3/2$ . Thus, Lemma 7 also holds. Then we proceed similarly to (29). In this case, for sufficiently large  $n$ , we obtain the density

$$\delta \leq \left( \frac{1}{2} + b \frac{\ln \ln n}{\ln n} + \frac{3}{4 \ln n} \right) \left( 1 + \frac{1}{\ln n} + \frac{1}{\ln^2 n} \right) \leq \frac{1}{2} + b \frac{\ln \ln n}{\ln n} + \frac{3}{2 \ln n}$$

which yields inequality (5).  $\square$

*Generalizations and concluding remarks.* Note that for  $n \rightarrow \infty$ , our choice of  $\beta > 1/2$  gives both an incremental expansion  $\theta_1/\theta_d \rightarrow 1$  in (29) and a vanishing fraction  $\omega$  in (30). By contrast, any constant  $\beta < 1/2$  increases the expansion ratio  $\theta_1/\theta_d$  to  $\exp(n^{1-2\beta})$ .

Our estimates can be slightly improved for finite  $n$ , by refining bounds (18) and (19). Also, we can employ the first bound in (17) instead of its approximation used throughout the paper. More precise estimates show that in this case, our bound (4) can be improved to

$$\delta \leq \frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{4 \ln \ln n}{\ln n}. \quad (31)$$

However, these refinements leave the asymptotic case  $n \rightarrow \infty$  unchanged.

The following Theorem can be proven by combining the technique of Theorem 1 with the multilayered design described in [4] and [5].

**Theorem 8** *A ball  $B_r^n$  of dimension  $n \rightarrow \infty$  and radius  $r_n \rightarrow \infty$  can be covered by the unit balls with density*

$$\delta(B_r^n) \leq \left( \frac{1}{2} + o(1) \right) n \ln n.$$

**Corollary 9** *Euclidean spaces  $\mathbf{R}^n$  of growing dimension  $n \rightarrow \infty$  can be covered by the unit balls with density*

$$\delta(\mathbf{R}^n) \leq \left( \frac{1}{2} + o(1) \right) n \ln n.$$

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